Integrality

Let R be a ring an S an R-algebra.

Def: $s \in S$ is integral over R if it is the zero of some monic polynomial in $R[x]$. If every element of S is integral over R, then S is integral over R.

We'll soon show that the set of elements integral over R is a subalgebra of S, called the integral closure of R in S (or the normalization of R in S). If R is an integral domain, the integral closure (or normalization) of R (w/out reference to an R-algebra) is the integral closure in its field of fractions.

Geometrically, normalizing a ring corresponds to "improving" the singularities of a variety. e.g. the normalization of a curve is always smooth!

Def: S is finite over R (or module-finite) if it is a finitely generated R-module (finite $\Rightarrow$ finitely generated as an R-algebra).

Ex: 1.) $R[x]$ is a f.g. R-algebra, but is not finite or integral over R.

2.) $R[x]/(x^2)$ is finite and integral over R.

3.) $\mathbb{Q}[\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \dots]$ is integral over $\mathbb{Q}$, but not finite.

Finiteness is a stronger condition than integrality:

Prop: S is finite over R iff S is generated as an R-algebra by finitely many integral elements. (i.e. $S = R[\alpha_1, \dots, \alpha_n]$, $\alpha_n \in S$)

image of R in S

generators integral over R

Pf: Suppose S is finite over R. If $a \in S$, consider the map $\varphi: S \longrightarrow S$ defined $s \longmapsto as$. Then C-H says φ satisfies $p(\varphi) = 0$ for some monic polynomial p. Thus $p(a)S = 0$ so $p(a) = 0$.

Conversely, suppose $S = R[\alpha_1, \dots, \alpha_n]$, α_i integral over R. Let $S' = R[\alpha_1, \dots, \alpha_{n-1}] \subseteq S$. By induction, we assume S' is finite over R, generated by $\{s_i\}$.
(finite)

α_n is integral over R and thus over S'. Let $p(x)$ be a monic polynomial over S' (or R) s.t. $p(\alpha_n) = 0$. Then we have a map

$$S'[x] \twoheadrightarrow S'[\alpha_n] = S$$
$$x \longmapsto \alpha_n$$

whose kernel contains p.

So $$S'[x]/(p) \twoheadrightarrow S \cong S'[x]/I$$

where I is the whole kernel of $S'[x] \to S$.

Thus, by the prop in the previous section, S is finite over S', generated by a finite set $\{t_i\}$. Thus, S is generated as an R-module by $\{s_i t_j\}$. (Exercise!) $\square$

Another application of C-H gives us a criterion for when an element is integral over a ring.

Prop: If S is an R-algebra and $s \in S$, then s is integral over R iff $\exists$ an S-module N and a f.g. R-submodule $M \subseteq N$ not annihilated by any nonzero element of S s.t. $sM \subseteq M$.

Cor: $s \in S$ is integral over $R \iff R[s]$ is finite over R.

Pf of Cor: ($\Rightarrow$) by previous prop.

($\Leftarrow$) Set $M = N = R[s]$, and apply prop. $\square$

Pf of prop: ($\Rightarrow$) Assume s is integral over R. Take $N = S$. Then $M = R[s] \subseteq S$. Thus, $M \cong R[x]/I$, where I contains some monic polynomial s satisfies, so M is finite over R.

($\Leftarrow$) Given $\underset{\substack{\uparrow \\ \text{finite} \\ \text{over } R}}{M} \subseteq \underset{\substack{\uparrow \\ S\text{-module}}}{N}$ modules as described, let $\varphi: M \to M$ be multiplication by s. Then we can apply C-H w/ $I = R$, and we get a monic polynomial $p(x) \in R[x]$ s.t. $p(s)M = 0$. But M is not annihilated by nonzero elts of S, so $p(s) = 0$. Thus, s is integral over R. $\square$

As mentioned earlier, integral elements over R in S form a subalgebra. In particular, if $s_1, \dots, s_n$ are integral over R, so is $R[s_1, \dots, s_n]$:

Thm: Let S be an R-algebra. The set of elements of S integral over R is a subalgebra of S.

Pf: Suppose $a, b \in S$ are integral over R. WTS $a-b$, ab are as well.

$R[a,b]$ is finite over R. Let $s = ab$ or $a-b$, $N = S$, $M = R[a,b]$. M is not annihilated by any element of S, and $sM \subseteq M$. Thus, s is integral over R (by prop). $\square$

Field extensions + Nullstellensatz revisited

We need to apply all of this to fields in order to finally finish the proof of the Nullstellensatz. Remember we need the following:

Thm: If k is algebraically closed, the maximal ideals of $k[x_1, \dots, x_n]$ are all of the form $(x_1 - a_1, \dots, x_n - a_n)$, $a_i \in k$.

Recall that we know all of these ideals are maximal. We just need to show these are exactly the maximal ideals. First we recall some field theory:

Suppose $K \subseteq L$ are fields, $v_1, \dots, v_n \in L$. Then $K(v_1, \dots, v_n)$ is the smallest subfield of L containing K and each v_i. Equivalently, it's the field of fractions of $K[v_1, \dots, v_n]$.

Def: L is a <u>finitely generated field extension</u> of K if $L = K(v_1, \dots, v_n)$ for some $v_1, \dots, v_n \in L$. L is an <u>algebraic extension</u> if all elements of L are algebraic over K.
i.e. satisfy a polynomial$/K$.

Ex: $\mathbb{Q}[\sqrt{5}] = \mathbb{Q}(\sqrt{5})$ since $\sqrt{5}\left(\frac{\sqrt{5}}{5}\right) = 1$. This is an algebraic extension of $\mathbb{Q}$. It's also finite over $\mathbb{Q}$ since it's generated by $1, \sqrt{5}$.

Check: If $K \subseteq L$ are fields, then the elements of L that are algebraic over K form a subfield.

Claim: $k(x)$ is not a finitely generated k-algebra.

Pf: Suppose $k(x) = k[v_1, \dots, v_n]$.

Then $\exists$ $b \in k[x]$ s.t. $bv_i \in k[x]$ for all v_i, and choose $c \in k[x]$ irreducible not dividing b.

Write $\frac{1}{c}$ as a k-linear combination of monomials in the v_i's. $\Rightarrow$ $\exists$ $N > 0$ s.t. $\frac{b^N}{c} \in k[x]$, a contradiction. $\square$

Claim: $k[x]$ is its own integral closure in $k(x)$.

Pf: Let $z \in k(x)$ be integral over $k[x]$. Then we have

$$z^n + a_{n-1} z^{n-1} + \dots + a_0 = 0, \quad a_i \in k[x].$$

If $z = \frac{f}{g}$, where $f, g \in k[x]$ are relatively prime, then multiplying through by g^n we get:

$$f^n + \underbrace{a_{n-1} f^{n-1} g + \dots + a_0 g^n}_{\text{divisible by } g} = 0 \Rightarrow g \mid f^n \Rightarrow g \in k. \ \square$$

We need one more lemma before proving the Nullstellensatz.

Lemma: Let $K \subseteq L$ be fields. If L is a f.g. K algebra, then L is finite (and thus algebraic) over K.

Pf: Let $L = K[v_1, \dots, v_n]$. We'll prove by induction on n.

If $n=1$, consider $K[x] \twoheadrightarrow K[v_1]$

$$x \longmapsto v_1$$

$K[v_1]$ is a field, so $K[v_1] \cong K[x]/(f)$, $f \neq 0$.

Thus, $f(v_1)=0 \Rightarrow v_1$ is algebraic over $K \Rightarrow K[v_1]$ is finite over K.

Now assume the statement holds for extensions generated by $n-1$ elements. Then $L = K(v_1)[v_2, \dots, v_n]$ is finite + algebraic over $K(v_1)$.

<u>Case 1</u>: v_1 algebraic $/K$. Then $K(v_1)$ is algebraic $/K$, so L is algebraic $/K$ by transitivity of integrality (HW #4).
So L is a K-algebra generated by finitely many integral elements, so it's finite over K.

<u>Case 2</u>: v_1 not algebraic over K. Then

$$K[x] \twoheadrightarrow K[v_1]$$

has 0 kernel, so $K(x) \cong K(v_1)$.

Each v_i satisfies some $v_i^n + a_1 v_i^{n-1} + \dots + a_n = 0$, $a_j \in K(v_1)$.

Set $a \in K[v_1]$ to be the product of the denominators of the a_j. Multiplying by a^n, we get

$$(av_i)^n + a a_1 (a v_i)^{n-1} + \cdots + a^n a_n = 0,$$

where all the coefficients are now in $K[v_1]$.

Thus, av_i is integral over $K[v_1]$.

Take $z \in L$ and $N >> 0$ s.t. $a^N z \in K[v_1][av_2, av_3, \dots, av_n]$, and is thus integral over $K[v_1]$.

Find $c \in K[v_1]$ relatively prime to a, and set

$$z = \frac{1}{c} \in K(v_1) \subseteq L.$$

Then for some $N > 0$, $\frac{a^N}{c} \in K(v_1)$ is integral over $K[v_1]$, so $\frac{a^N}{c} \in K[v_1]$, which is a contradiction since a and c are rel. prime. $\square$

Now we finish the Nullstellensatz:

Theorem: If $k = \bar{k}$, and $m \subseteq k[x_1, \dots, x_n] = R$ is maximal, then

$$m = (x_1 - a_1, \dots, x_n - a_n), \text{ some } a_i \in k.$$

Pf: let $L = R/m$. L is a field and $k \subseteq L$.

L is f.g. over k, so it's algebraic over k. i.e. if $z \in L$, then $f(z) = 0$, some $f \in k[x]$.

k is algebraically closed, so $z \in k$. Thus $L = k$.

Thus, for all x_i, there is some $a_i \in k$ s.t. $\overline{x_i} = \overline{a_i}$ in L

$\Rightarrow x_i - a_i \in \mathfrak{m}$.

$\Rightarrow (x_1 - a_1, \dots, x_n - a_n) \subseteq \mathfrak{m}$, but, is maximal, so they are equal. $\square$